Besides, we also want to take this chance to introduce some new concepts:

- exponential map ← new
- curvature tensor ← generalization of Riem 2-tensor
- principal bundle (主丛) ← generalization of vector bundle
- representation

Due to time limit, we can only touch a bit for each one bullet above.

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1. Tangent space of G (Lie algebra).

Notation: $e \in G$ unit 单位元

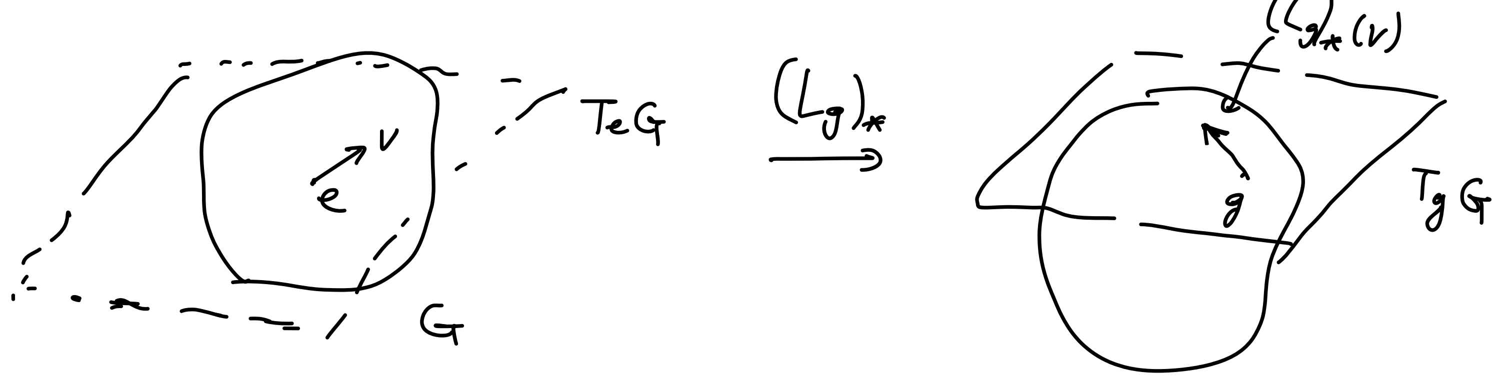
$g \cdot G \to G$ left multiplication by g. denote by L_g.

Def Fix any $v \in T_e G$, define a vector field on G by

$$X^v(g) := (L_g)_*(e)(v) \quad (\text{or } (dL_g)_e(v))$$

at $T_g G$

(recall: $(L_g)_* : T_e G \longrightarrow T_g G$).



Observation on X^v:

- $\forall h \in G$, consider $L_h : G \longrightarrow G$, then for X^v, at $g \in G$

$$\underbrace{(L_h)_*(g)\,(X^v(g))}_{\in T_{h \cdot g} G} = (L_h)_*(g)\,(L_g)_*(e)\,(v)$$

$$= (L_h \cdot L_g)_*(e)\,(v)$$

$$= (L_{h\cdot g})_*(e)(v) = X^v(h\cdot g)$$

In other words, $(L_h)_* X^v = X^v$ (for any $h \in G$).

Therefore, X^v defined by Def above is also called a <u>left invariant vector field</u> (generated by $v \in T_eG$)

<u>Rmk</u> One can define a vector field $X \in \Gamma(TG)$ left invariant if it satisfies $(L_h)_*(g)(X(g)) = X(h\cdot g)$ $\forall h, g \in G$. However, this definition is redundant, since any left invariant vector field X must be in the form $X = X^v$ for some $v \in T_eG$. b/c

$$X(g) = (L_g)_*(e)(X(e)) \quad \forall g.$$

i.e. X is uniquely determined by $X(e)$ $(=: v)$.

$$\Longrightarrow \{\text{left invariant v.f. on } G\} \overset{1:1}{\longleftrightarrow} \{v \in T_eG\}$$

Denote by $\mathfrak{g} := \{$ left inv. v.fs on $G\} = T_eG$ $\left(\simeq \mathbb{R}^{\dim G} \text{ or } \mathbb{C}^{\dim_{\mathbb{C}} G}\right)$

$\mathfrak{g}$: vector space

Interestingly, there exist an additional structure on $\mathfrak{g}$ (on T_eG).

<u>Prop</u> If X, Y are left inv. v.fs on G, then $[X, Y]$ is also a left inv. v.f on G.

<u>Pf</u>: By Midterm Exam $(L_h)_*([X,Y]) = [(L_h)_*(X), (L_h)_*(Y)] = [X, Y]$. □

$\Rightarrow$ $X = X^v$ where $v = X(e)$

$Y = Y^w$ where $w = Y(e)$.

$\rightsquigarrow$ $[X, Y] = [X, Y]^u$ when $u = [X,Y](e)$.

Then define $[\ ,\ ]$ on $\mathfrak{g}$ by

$$[v, w] := u$$

where v, w, u are given as above

Def – A Lie algebra (over $\mathbb{R}$ or $\mathbb{C}$) is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) V equipped with a bilinear operation $[-,-]: V \times V \to V$ satisfying

- $[v,w] = -[w,v] \iff [x,x] = 0 \;\forall x \in V$
- $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$ ← Jacobi identity

– A Lie subalgebra is a linear subspace $W \subseteq V$ s.t. $[\,,\,]$ is closed on W, i.e. $[w_1, w_2] \in W$ if $w_1, w_2 \in W$.

($\Rightarrow$ $(W, [\,,\,]|_W)$ itself is a Lie algebra).

– A Lie algebra homomorphism $F: (W, [\,,\,]_W) \longrightarrow (V, [\,,\,]_V)$ is a linear map s.t. $[F(w_1), F(w_2)]_V = F([w_1, w_2]_W)$

Rmk ① $\ker(F)$ and $\mathrm{im}(F)$ are Lie subalgebras

② A Lie algebra isomorphism is a Lie alg homomorphism + bijective.

One should view $[\,,]$ defines a "product str." (anti-symmetric) on V, so a Lie algebra combines linear algebra and group theory.

(cf. Lie group combines smooth mfd and group theory.)

Example

① For any mfd M, $\Gamma(TM) = \{$ vector fields on $M\}$ is a Lie algebra, where $[\,,]$ is the bracket of vector fields. ← depending on the connection chosen on M

② For a Lie group G, $\{$ left inv. vector fields on $G\}$ $(=: \mathfrak{g})$ is a Lie subalgebra of $\Gamma(TM)$, w.r.t bracket $[\,,]$.

③ $M_{n\times n}(\mathbb{R}) = \{$ $n\times n$ matrices over $\mathbb{R}\}$ $(\simeq \mathbb{R}^{n^2})$ by

$$A = (a_{ij})_{1\le i,j\le n} \longleftrightarrow (a_{11}, a_{12}, \cdots, a_{nn})$$
$$\overset{\shortparallel}{a_1}$$

Define $[A, B] := A \cdot B - B \cdot A$.

Then $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ ← by directly expanding out all terms.

$\Rightarrow$ one can equip $\mathbb{R}^{n^2}$ with a Lie algebra str. by identify $a \in \mathbb{R}^{n^2}$ with a matrix $A(a) \in M_{n \times n}(\mathbb{R})$, then

$$[a, b] := [A(a), A(b)] \longleftrightarrow \text{vector in } \mathbb{R}^{n^2}.$$

Rmk Often one can equip any vector space V a Lie algebra str. by assigning $[\,,\,] \equiv 0$. This is called the abelian Lie algebra str.

④ Give a Lie group G,

$$\mathfrak{g} = T_e G \quad (= \{\text{left inv. vector fields on } G\})$$

is a Lie algebra. $[v, w] := [X^v, X^w](e)$.

Usually, $\mathfrak{g}$ (or $\mathfrak{g}_G$) is called the Lie algebra of G.

Computing Lie algebra $\mathfrak{g}_G$ of a Lie group G is a basic knowledge in smooth mfd.

<u>Example</u> ① $(\mathbb{R}^n, +)$ Lie group. and left multiplication L_g is

$$\underset{\mathbb{R}^n}{\underset{\cap}{L_g}}(x) = x + g \quad \Longrightarrow \quad (L_g)_*(\underset{\underset{0 \in \mathbb{R}^n}{\|}}{e}) = \mathbb{1}$$

$\leftarrow$ identity $n \times n$ matrix

Then for any $v \in T_0 \mathbb{R}^n \ (= \mathbb{R}^n)$, we have

$$X^v(g) = (L_g)_*(e)(v) = \mathbb{1} \cdot v = v.$$

$\swarrow$ This is a constant vector field.

$$\Longrightarrow \quad [X^v, X^w] = 0$$

So $(\mathfrak{g}_{\mathbb{R}^n}, [\,,\,]) = (\mathbb{R}^n, [\,,\,] \equiv 0)$, an abelian Lie alg. str.

② $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$ Lie group str. $(\theta_1, \cdots, \theta_n) \cdot (\sigma_1, \cdots \sigma_n) = (\underset{\text{mod } 1}{\theta_1 + \sigma_1}, \cdots, \underset{\text{mod } 1}{\theta_n + \sigma_n})$

In particular, $L_g(x) = x+g \pmod 1 \Rightarrow \underset{(0,\dots,0)\in \mathbb{T}^n}{(L_g)_*(e)} = \mathbb{1}$

Then similarly as above, $(\mathfrak{g}_{\mathbb{T}^n}, [\,,\,]) = (\mathbb{R}^n, [\,,\,]\equiv 0)$.

<u>Rmk</u> Simply from Lie algebra, one can't tell what the Lie group is. (Later, we will see a more surprising result.)

*Slogan: Lie algebra $\mathfrak{g}_G$ inherit (继承) many properties from Lie group G.

e.g. $GL(n,\mathbb{R})$ $(\subseteq M_{n\times n}(\mathbb{R}))$ a Lie group with matrix multiplication.

compute its Lie algebra $\mathfrak{g}_{GL(n,\mathbb{R})} = T_{\mathbb{1}}GL(n,\mathbb{R})$ $(= \mathbb{R}^{n^2})$.

viewed as a matrix

For $v = (v_{ij}) \in T_{\mathbb{1}}(GL(n,\mathbb{R}))$

basis of $\mathbb{R}^{n^2}$ (double index)

$$X^v(A) = \underbrace{(L_A)_*(\mathbb{1})(v)}_{\in T_A GL(n,\mathbb{R})} = \lim_{t\to 0} \frac{A(\mathbb{1}+tv) - A}{t} = A\cdot v$$

$$= \sum_{ij}\left(\sum_{k=1}^n A_{ik}v_{kj}\right)\frac{\partial}{\partial x_{ij}}$$

Then for $v, w \in T_{\mathbb{1}}(GL(n,\mathbb{R}))$, we have

$$[X^v, X^w] = \left[\sum_{ij}\left(\sum_{k=1}^n A_{ik}V_{kj}\right)\frac{\partial}{\partial x_{ij}},\ \sum_{pq}\left(\sum_{r=1}^n A_{pr}W_{rq}\right)\frac{\partial}{\partial x_{pq}}\right]$$

$$= \sum_{iq}\left(\sum_{k,j=1}^n A_{ik}V_{kj}W_{jq}\right)\frac{\partial}{\partial x_{iq}} - \sum_{pj}\left(\sum_{r,q=1}^n A_{pr}W_{rq}V_{qj}\right)\frac{\partial}{\partial x_{pj}}$$

switch j, q
redenote p
by i

$$= \sum_{ij}\left(\sum_{k,q} A_{ik}\left(V_{kq}W_{qj} - W_{kq}V_{qj}\right)\right)\frac{\partial}{\partial x_{ij}}$$

Then evaluate $[X^v, X^w]$ at $\mathbb{1}$ $\left(\Leftrightarrow A_{ik} = \begin{cases} 1 & i=k \\ 0 & i\neq k\end{cases}\right)$

$$[v,w] = [X^v, X^w](\mathbb{1})$$

$$= \sum_{ij}\left(\sum_q V_{iq}W_{qj} - W_{iq}V_{qj}\right)\frac{\partial}{\partial x_{ij}} \rightsquigarrow \text{matrix} \in M_{n\times n}(\mathbb{R}) \text{ by } vw - wv.$$

Summary: Lie algebra str. on $T_{\mathbb{1}}GL(n,\mathbb{R})$ is the commutator of $GL(n,\mathbb{R})$.
standard

Notation: the Lie algebra of Lie group $GL(n,\mathbb{R})$ is denoted by

$$\mathfrak{gl}(n,\mathbb{R}) \quad \left(= \left(\mathbb{R}^{n^2}, [\,,\,]\right) \right)$$

← matrix commutator.

<u>Exe.</u> Prove/compute the following Lie algebra

$$\mathfrak{g}_{SL(n,\mathbb{R})} =: \mathfrak{sl}(n,\mathbb{R}) = \{ A \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{tr} A = 0 \}$$

$$\mathfrak{g}_{O(n)} =: \mathfrak{o}(n) = \{ A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^T + A = 0 \}$$

s.t. $\mathbb{C}^2$

$$\mathfrak{g}_{SL(n,\mathbb{C})} =: \mathfrak{sl}(n,\mathbb{C}) = \{ A \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{tr} A = 0 \}$$

← transpose + conjugate

$$\mathfrak{g}_{U(n)} =: \mathfrak{u}(n) = \{ A \in \mathfrak{gl}(n,\mathbb{C}) \mid A^* + A = 0 \}$$

$$\mathfrak{g}_{Sp(2n)} =: \mathfrak{sp}(2n) = \{ A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^T J + JA = 0 \}$$

$$J = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}$$

e.g. If G is an abelian Lie group, then its Lie algebra $\mathfrak{g}_G$ admits an abelian Lie str. (i.e. $[\,,\,] \equiv 0$).

Pf Consider inverse map $inv: G \to G$ by $g \mapsto g^{-1}$.

G is an abelian group $\Longrightarrow$ inv is a Lie group homomorphism (isomorphism)

b/c $inv(gh) = inv(hg) = (hg)^{-1} = g^{-1} \cdot h^{-1} = inv(g) \cdot inv(h)$.

Then the pushforward $(inv)_*(e): \underset{\mathfrak{g}}{T_eG} \to \underset{\mathfrak{g}}{T_eG}$ (b/c $inv(e) = e$)

and explicitly $v \in T_eG = \mathfrak{g}$

$$(inv)_*(e)(v) = \lim_{t \to 0} \frac{(e+tv)^{-1} - e}{t} = \frac{(e - tv + o(t)) - e}{t} = -v$$

Therefore, $\forall\, v, w \in T_eG = \mathfrak{g}$

$$\begin{aligned} -[v,w] &= (inv)_*(e)([v,w]) \\ &= (inv)_*(e)([X^v, X^w](e)) = [inv_*(X^v), inv_*(X^w)](e) \\ &= [-X^v, -X^w](e) = [v,w] \Longrightarrow [v,w] = 0 \ \square \end{aligned}$$

e.g. If $\varphi: G \to H$ is a Lie group homomorphism, then

$$(d\varphi)(e): \mathfrak{g}_G (= T_{e_G} G) \to \mathfrak{g}_H (= T_{e_H} H) \text{ is a Lie algebra homomorphism.}$$

Exe Prove the claim above.

In a categorical language: Lie = cat of Lie groups ← Mor = Lie group homomorphism

lie = cat of Lie algebras ← Mor = Lie algebra homomorphism

Then $\varphi \to (d\varphi)(e)$ is a functor from Lie to lie.

As a concrete example: consider for fixed $g \in G$,

$$c(g): G \to G \quad \text{by} \quad c(g) = L_g \circ R_{g^{-1}}.$$

(so $c(g)(x) = gxg^{-1}$). This is a Lie group homomorphism (iso.)

Then $d\,c(g)(e): \mathfrak{g}_G \to \mathfrak{g}_G$ is a Lie algebra homomorphism.

Take $G = GL(n, \mathbb{R})$, let us compute $dc(g)$ explicitly.

$$dc(g)(e)\underset{\substack{\cap \\ gl(n,\mathbb{R})}}{(A)} = \frac{d}{dt}\Big|_{t=0}\Big(g(e+tA)g^{-1} - e\Big) = g\cdot A\cdot g^{-1}$$

matrix multiplication.

This is a linear transformation (for each $g \in GL(n,\mathbb{R})$) on vectorspace $gl(n,\mathbb{R})$.

Def A Lie group representation (of a given Lie group G) is a Lie group homomorphism $\rho: G \longrightarrow GL(V)$ for some vector space V.

$GL(V)$ = collection of linear transformation of V

e.g. Given a Lie group G, via $c(-)$ as above, one can construct a Lie group rep. $\rho := c(-): G \longrightarrow GL(\mathfrak{g})$

$$g \longmapsto \rho(g) = dc(g)(e)$$

it is a homo by the chain rule of pushforward.

This representation is called the adjoint representation of G, denoted by Ad_g $(= dc(g)(e))$.

e.g. Euclidean group $E(n) := \mathbb{R}^n \rtimes O(n)$ where the product str.

semidirect production

is $(v, A)\cdot(w, B) = (v + Aw, AB)$.

(where $E(n)$ acts on $\mathbb{R}^n$ by $(v, A)\cdot x = Ax + v$)

shift

rotation (including reflection)

<u>Rmk</u> Euclidean group is the isometry group of Euclidean space w.r.t the distance metric.

$E(n)$ is a Lie group (b/c semi-direct product of two Lie groups is a Lie group.)

Consider $\rho : E(n) \longrightarrow GL(\mathbb{R}^{n+1})$ $(= GL(n+1;\mathbb{R}))$ by

$$\rho((v, A)) = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}$$

Then $\rho((v,A)\cdot(w,B)) = \begin{pmatrix} AB & v+Aw \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B & w \\ 0 & 1 \end{pmatrix} = \rho((v,A))\cdot\rho((w,B))$.

Reference recommended: Savage, Quivers and the Euclidean group.

<u>Rmk</u> A (Lie group) rep. is called <u>faithful</u> if it is injective.

e.g. For $GL(n,\mathbb{R})$, the adjoint rep $Ad_g(-) = g\cdot(-)\cdot g^{-1}$ is faithful.
(In general, adjoint rep may not be faithful.)

e.g. $\rho: E(n) \to GL(n+1,\mathbb{R})$ is faithful.

Back to the adjoint rep $Ad: G \to GL(\mathfrak{g})$ $\left(\cong GL(\mathbb{R}^{\dim G})\right)$, observe that both G and $GL(\mathfrak{g})$ are Lie group (again) and Ad is a Lie group homomorphism, then passing to the pushforward:

$$d(Ad)(e) : T_eG = \mathfrak{g} \longrightarrow T_1 GL(\mathfrak{g}) =: \mathfrak{gl}(\mathfrak{g}).$$

linear transformations of vector space $\mathfrak{g}$.

This is a Lie <u>algebra</u> homomorphism.

Let us compute $d(Ad)(e)$:

for any $v \in \mathfrak{g}$,

a linear transformation (or a matrix)

$$d(\mathrm{Ad})(e)(v) = \frac{d}{dt}\Big|_{t=0} \frac{\mathrm{Ad}(e+tv) - \mathrm{Ad}(e)}{t}$$ ← identity map

Step here is not completely satisfying since $e+tv$ does not make sense in a general group

$$= \frac{d}{dt}\Big|_{t=0} \frac{d(c(e+tv)) - d(c(e))}{t}$$ ← identity

$$= v\cdot(-) - (-)\cdot v$$

In other words, $\big(d(\mathrm{Ad})(e)(v)\big)(w) = [v, w]$

One usually denote $d(\mathrm{Ad})(e)$ by ad (so we get $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$)

This is called the adjoint representation <u>of Lie algebra</u> $\mathfrak{g}$.

(In general, one can define an adjoint rep of a Lie algebra $\mathfrak{g}$ as a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for some vector space V.

viewed as linear transformation equipped with $[\,,\,]$ as matrix commutator.)

Recall Ado-Iwasawa's Thm in earlier notes (9/29/2024 first notes, page 1)

It roughly says a real Lie group G can embed into $GL(n, \mathbb{R})$.

The reality is:

<u>Thm (Ado-Iwasawa)</u> Every f.d. real Lie algebra $\mathfrak{g}$ admits a faithful finite-dim'l representation, i.e. $\exists n$, s.t. $\mathfrak{g} \xrightarrow{\rho} \mathfrak{gl}(\mathbb{R}^n)$.

(Related to a Lie group G, one can take $\mathfrak{g} = \mathfrak{g}_G$).

$\Rightarrow$ Any f.d. real Lie algebra $\mathfrak{g} \simeq$ a Lie subalgebra of $\mathfrak{gl}(\mathbb{R}^n)$. cf Whitney embedding Thm.

<u>Rmk</u> Interestingly, <u>not</u> every f.d. real Lie group "embeds" into $GL(n, \mathbb{R})$ via a Lie group representation (e.g. $\widetilde{SL(2,\mathbb{R})}$, see Exe 21-26 in Lee's fat book).

<u>Question</u>: $G \underset{\text{seen}}{\rightsquigarrow} \mathfrak{g}_G$ how about $\mathfrak{g}_G \rightsquigarrow G$?